SEMIGROUP COMPACTIFICATIONS IN TERMS OF FILTERS

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Abstract. We present a study of semigroup compactifications of a semitopological semigroup S in terms of filters. We characterize closed subsemigroups and closed left, right, and two-sided ideals of any semigroup compactification of S in terms of filters and in terms of ideals of the corresponding m-admissible subalgebra of $C(S)$. Under some conditions on the semigroup compactification in question, we characterize the points in the minimal ideal or in its closure for any semigroup compactification of S in which S is embedded.

1. Introduction

Filters have proven to be an extremely powerful tool in analyzing the algebraic properties of the Stone-Čech compactification βS of a discrete semigroup S. Indeed, when the algebra of βS is considered, it is customary to view βS as the space of all ultrafilters on S (see [5]). In more general context, the semigroup compactifications of a semitopological semigroup S appear as the spectra of m-admissible subalgebras of $C(S)$. A representation of the spectrum of any C^*-subalgebra $\mathcal{F}$ of $\ell^\infty(X)$ containing the constant functions, where X is any non-empty set, was established by the author in [2] using $\mathcal{F}$-ultrafilters. In the paper, we apply this representation to study semigroup compactifications.

In Section 3, we describe the semigroup operation of any semigroup compactification of a semitopological semigroup S in terms of $\mathcal{F}$-filters, where $\mathcal{F}$ is the m-admissible subalgebra of $C(S)$ corresponding to the semigroup compactification in question. It is worth noticing that this description is similar to the description of the semigroup operation of βS for a discrete semigroup S as given in [5]. In Section 4, we describe closed subsemigroups and closed left, right, and two-sided ideals of any semigroup compactification of S both in terms of $\mathcal{F}$-filters and the corresponding ideals of $\mathcal{F}$. In the last section,

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we describe the points of the minimal ideal and its closure under some conditions on the semigroup compactification. These conditions are satisfied for a discrete semigroup and its Stone-Čech compactification, or if S is algebraically a group and S is embedded in the semigroup compactification in question.

The paper is not the first one dealing with filter properties of semigroup compactifications. However, as far as the author is aware, the results presented in Sections 3 and 4 are the first ones including all the semigroup compactifications of a semitopological semigroup actually using filters. The $\mathcal{WAP}$-compactification of a discrete semigroup is studied using filters in [3]. Near ultrafilters were applied in [6] to study the $\mathcal{LUC}$-compactification of a topological group, and $\mathcal{LUC}$-ultrafilters were applied by the author in [1] for the same purposes. Any semigroup compactification of a semitopological semigroup is studied in terms of certain equivalence classes of z-ultrafilters in [7] and [8]. Analogues of the results given in Section 5 are given in [7] for the $\mathcal{LMC}$-compactification of a locally compact semitopological semigroup.

2. Preliminaries

We recall briefly some definitions and introduce some notation that we will use throughout the paper.

If X is a Hausdorff topological space, then we denote by $C(X)$ the C^*-algebra of all bounded, continuous, complex-valued functions on X. If X is also locally compact, then we denote by $C_0(X)$ the C^*-subalgebra of $C(X)$ consisting of those elements of $C(X)$ which vanish at infinity.

Let S be a semigroup and let $s \in S$. The functions $\lambda_s : S \to S$ and $\rho_s : S \to S$ are defined by $\lambda_s(t) = st$ and $\rho_s(t) = ts$ for every $t \in S$, respectively. Let $A \subseteq S$. If there is no danger of confusion, then we denote $\lambda_s^{-1}(A)$ and $\rho_s^{-1}(A)$ by $s^{-1}A$ and As^{-1}, respectively. If $f : S \to \mathbb{C}$ is any function and $r > 0$, then we define

$$Z(f) = \{s \in S : f(s) = 0\} \quad \text{and} \quad X(f, r) = \{s \in S : |f(s)| \leq r\}.$$

Also, for every $s \in S$, the functions $L_s f : S \to \mathbb{C}$ and $R_s f : S \to \mathbb{C}$ are defined by $L_s f = f \circ \lambda_s$ and $R_s f = f \circ \rho_s$, respectively.

A *right topological semigroup* is a semigroup S endowed with a Hausdorff topology such that $\rho_s : S \to S$ is continuous for every $s \in S$. The *topological centre* of a right topological semigroup S is the set $\Lambda(S) = \{s \in S : \lambda_s : S \to S \text{ is continuous}\}$. A *semitopological semigroup* is a right topological semigroup S such that $\lambda_s : S \to S$ is continuous for every $s \in S$.

Let S be a semitopological semigroup and let $\mathcal{F}$ be a C^*-subalgebra of $C(S)$ containing the constant functions. By an ideal of $\mathcal{F}$, we always mean a closed, proper ideal of $\mathcal{F}$. We denote by $\tau(\mathcal{F})$ the weak topology on S generated by $\mathcal{F}$ and we denote by A° the $\tau(\mathcal{F})$-interior of A. We consider the spectrum Δ of $\mathcal{F}$ as the space of all multiplicative, linear functionals on $\mathcal{F}$. The evaluation mapping $\varepsilon : S \to \Delta$ is given by $[\varepsilon(s)](f) = f(s)$ for every $s \in S$ and for every $f \in \mathcal{F}$. We recall briefly the consideration of the spectrum Δ as the space of certain filters on S. For more details, see [2].

An $\mathcal{F}$-*family* on S is a non-empty set $\mathcal{A}$ of non-empty subsets of S such that, for every set $A \in \mathcal{A}$ with $A \neq S$, there exist a set $B \in \mathcal{A}$ and a function $f \in \mathcal{F}$ such that $f(B) = \{1\}$ and $f(S \setminus A) = \{0\}$. An $\mathcal{F}$-*filter* on S is a filter φ on S such that φ is an $\mathcal{F}$-family on S. An $\mathcal{F}$-*ultrafilter* on S is an $\mathcal{F}$-filter on S which is not properly contained in any other $\mathcal{F}$-filter.

If I is an ideal of $\mathcal{F}$, then the family

$$\mathcal{B}(I) = \{X(f, r) : f \in I,\, r > 0\}$$

is a filter base on S and the filter on S generated by $\mathcal{B}(I)$ is an $\mathcal{F}$-filter. Conversely, if φ is an $\mathcal{F}$-filter on S, then there exists a unique ideal I of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$.

Let δS be the set of all $\mathcal{F}$-ultrafilters on S. For every subset A of S, define $\widehat{A} = \{p \in \delta S : A \in p\}$. We equip δS with the topology which has the family $\{\widehat{A} : A \subseteq S\}$ as its base. Then the mapping $\mu \mapsto p_\mu$, where p_μ is the $\mathcal{F}$-ultrafilter on S generated by $\mathcal{B}(\ker \mu)$, from Δ to δS is a homeomorphism. If Y is a subset of δS, then we denote by $\overline{Y}$ the closure of Y in δS with one exception: If $A \subseteq S$, then we use $\mathrm{cl}_{\delta S}(\widehat{A})$ instead of $\overline{\widehat{A}}$. If $f \in \mathcal{F}$, then there exists a unique function $\widehat{f} \in C(\delta S)$ such that $f = \widehat{f} \circ e$. Furthermore, the mapping $f \mapsto \widehat{f}$ from $\mathcal{F}$ to $C(\delta S)$ is an isometric $*$-isomorphism.

The *evaluation mapping* $e : S \to \delta S$ is defined as follows. For every $s \in S$, let $\mathcal{N}_{\mathcal{F}}(s)$ be the neighborhood filter of s in the weak topology generated by $\mathcal{F}$. Then $\mathcal{N}_{\mathcal{F}}(s)$ is an $\mathcal{F}$-ultrafilter on S and the evaluation mapping is defined by $e(s) = \mathcal{N}_{\mathcal{F}}(s)$ for every $s \in S$.

Let φ be an $\mathcal{F}$-filter on S. If $A \subseteq S$, then $A \in \varphi$ if and only if $A^\circ \in \varphi$. In particular, $\widehat{A} = \widehat{A^\circ}$ for every subset A of S. Define

$$\widehat{\varphi} = \{p \in \delta S : \varphi \subseteq p\} \quad \text{and} \quad \overline{\varphi} = \bigcap_{A \in \varphi} \overline{e(A)}.$$

Then $\widehat{\varphi} = \overline{\varphi}$. Also, $\overline{\varphi}$ is a non-empty, closed subset of δS and, conversely, for every non-empty, closed subset C of δS, there exists a unique $\mathcal{F}$-filter φ on S such that $C = \overline{\varphi}$.

3. Description of the semigroup operation of δS

For the rest of the paper, let S be a semitopological semigroup and let $\mathcal{F}$ be an m-admissible subalgebra ([4, p. 77]) of $C(S)$. Since the mapping $\mu \mapsto p_\mu$ is a homeomorphism and $\varepsilon(s) \mapsto e(s)$ for every $s \in S$, the space δS is a semigroup compactification of S ([4, pp. 105, 108]). In this section, we describe the semigroup operation of δS.

Definition 1. Let φ be an $\mathcal{F}$-filter on S and let $A \subseteq S$. Define

$$\Omega_\varphi(A) = \{s \in S : s^{-1}A \in \varphi\}.$$

Statements (i)-(iii) of the next lemma follow immediately from the previous definition and statement (iv) follows from [2, Theorem 2 (ii)].

Lemma 1. Let φ and ψ be $\mathcal{F}$-filter on S and let $A, B \subseteq S$. The following statements hold:

(i) If $A \subseteq B$, then $\Omega_\varphi(A) \subseteq \Omega_\varphi(B)$.
(ii) If $\varphi \subseteq \psi$, then $\Omega_\varphi(A) \subseteq \Omega_\psi(A)$.
(iii) $\Omega_\varphi(A \cap B) = \Omega_\varphi(A) \cap \Omega_\varphi(B)$.
(iv) $\Omega_\varphi(A) = \bigcap_{p \in \overline{\varphi}} \Omega_p(A)$.

In the next lemma, we need to consider several cases while describing the semigroup operation of δS. Unlike with the Stone-Čech compactification of a discrete semigroup, the subset $\widehat{A}$ of δS for some subset A of S need not be closed. However, if S is discrete and $\mathcal{F} = C(S)$, then our results below agree with the usual properties of the semigroup operation of the Stone-Čech compactification of S (see [5, p. 76]).

Lemma 2. Let $A \subseteq S$, let $s \in S$, and let $p, q \in \delta S$. The following statements hold:

(i) If $A \in e(s)q$, then $s^{-1}A \in q$.
(ii) If A is $\tau(\mathcal{F})$-open and $q \in \mathrm{cl}_{\delta S}(\widehat{s^{-1}A})$, then $e(s)q \in \mathrm{cl}_{\delta S}(\widehat{A})$.
(iii) If $A \in pq$, then $\Omega_q(A) \in p$.
(iv) If $p \in \mathrm{cl}_{\delta S}(\widehat{\Omega_q(A)})$, then $pq \in \mathrm{cl}_{\delta S}(\widehat{A})$.

Proof. (i) Suppose that $A \in e(s)q$. Since $\lambda_{e(s)}$ is continuous on δS, there exists a $\tau(\mathcal{F})$-open subset B of S such that $B \in q$ and $\lambda_{e(s)}(\widehat{B}) \subseteq \widehat{A}$. Then $e(sx) \in \widehat{A}$ for every $x \in B$, so $sx \in A$ for every $x \in B$, and so $B \subseteq s^{-1}A$. Therefore, $s^{-1}A \in q$.

(ii) Suppose that A is a $\tau(\mathcal{F})$-open subset of S and that $q \in \widehat{s^{-1}A}$. If $B \in e(s)q$, then $\lambda_s^{-1}(B^\circ) \in q$ by statement (i), so $\lambda_s^{-1}(B^\circ) \cap s^{-1}A \neq \emptyset$, and so $B^\circ \cap A \neq \emptyset$. Therefore, $\widehat{B} \cap \widehat{A} \neq \emptyset$, as required.

(iii) Suppose that $A \in pq$. Since ρ_q is continuous on δS, there exists a $\tau(\mathcal{F})$-open subset B of S such that $B \in p$ and $\rho_q(\widehat{B}) \subseteq \widehat{A^\circ}$. Then $e(s)q \in \widehat{A^\circ}$ for every $s \in B$, and so $B \subseteq \Omega_q(A)$ by statement (i). Therefore, $\Omega_q(A) \in p$.

(iv) Suppose that $pq \notin \mathrm{cl}_{\delta S}(\widehat{A})$. Then there exists a $\tau(\mathcal{F})$-open subset B of S such that $B \in pq$ and $\widehat{B} \cap \widehat{A^\circ} = \emptyset$, and so $A^\circ \cap B = \emptyset$. Now, $\Omega_q(B) \in p$ by statement (iii). By Lemma 1 (iii), we have $\Omega_q(A) \cap \Omega_q(B) = \emptyset$, and so $\widehat{\Omega_q(A)} \cap \widehat{\Omega_q(B)} = \emptyset$. Therefore, $p \notin \mathrm{cl}_{\delta S}(\widehat{\Omega_q(A)})$. □

We leave the details of the next corollary to the reader.

Corollary 1. Let $p, q \in \delta S$ and let $A \subseteq S$. The following statements hold:

(i) If $A \in pq$, then there exist a set $B \in p$ and a family $\{C_s : s \in B\}$ of members of q such that $\bigcup_{s \in B} sC_s \subseteq A$.
(ii) If there exist a set $B \in p$ and a family $\{C_s : s \in B\}$ of members of q such that $\bigcup_{s \in B} sC_s \subseteq A$, then $pq \in \mathrm{cl}_{\delta S}(\widehat{A})$.

We finish this section with some remarks concerning the previous lemma. First, statement (ii) does not hold for arbitrary subsets of S. For example, consider the multiplicative semigroup $S = [0, \infty[$ with the Euclidean topology and let $\mathcal{F}$ be any m-admissible subalgebra of $C(S)$. Put $s = 0$ and $A = \{0\}$. Then $s^{-1}A = S$, and so $s^{-1}A \in q$ for every $q \in \delta S$. Since $A^\circ = \emptyset$, we have $\widehat{A} = \emptyset$. However, if S is algebraically a group and A is any subset of S, then $A \in e(s)q$ if and only if $s^{-1}A \in q$.

The closure $\mathrm{cl}_{\delta S}(\widehat{A})$ in statement (ii) is essential. That is, $A \in e(s)q$ need not hold if A is a $\tau(\mathcal{F})$-open subset of S and $q \in \mathrm{cl}_{\delta S}(\widehat{s^{-1}A})$. Again, let $S = [0, \infty[$ be equipped with the Euclidean topology but let the semigroup operation on S be given by $st = \min\{s, t\}$. Let $\mathcal{F} = C_0(S) \oplus \mathbb{C}$, where $\mathbb{C}$ denotes the constant functions on S. It is easy to verify that $\mathcal{F}$ is an m-admissible subalgebra of $C(S)$ and that δS is the one-point compactification of S with ∞ as its identity. Let $s > 0$ and put $A =]s, \infty[$. Then A is $\tau(\mathcal{F})$-open in S. Since $s^{-1}A = A$ and $S \setminus A$ is a compact subset of S, we have $s^{-1}A \in \infty$. However, $s \notin A$, and so $A \notin e(s) = e(s)\infty$.

4. Closed subsets of δS

This section is devoted to a characterization of closed subsemigroups and closed left, right, and two-sided ideals of δS in terms of $\mathcal{F}$-filters and the corresponding ideals of $\mathcal{F}$. Recall that the *left introversion operator* determined by an element $p \in \delta S$ is defined by

$$(T_p f)(s) = \widehat{L_s f}(p)$$

for every $f \in \mathcal{F}$ and for every $s \in S$.

We leave the verification of the next lemma to the reader.

Lemma 3. If $f \in \mathcal{F}$, $s \in S$, and $p \in \delta S$, then $\widehat{L_s f} = L_{e(s)} \widehat{f}$ and $\widehat{T_p f} = \rho_p \widehat{f}$.

Lemma 4. If $f \in \mathcal{F}$, $p \in \delta S$, and $r > 0$, then

$$X(T_p f, r/2) \subseteq \Omega_p(X(f,r)) \subseteq X(T_p f, r).$$

Proof. Suppose that $s \in X(T_p f, r/2)$. Put $B = \{t \in S : |f(t)| < r\}$. Now,

$$A := \{u \in S : |(L_s f)(u)| < r\} = s^{-1}B \subseteq \lambda_s^{-1}(X(f,r)^\circ).$$

Since $A \in p$ by [2, Lemma 6], we have $s \in \Omega_p(X(f,r))$, thus proving the first inclusion of the lemma. If $s \in \Omega_p(X(f,r))$, then $\lambda_s^{-1}(X(f,r)) \in p$, so $X(L_s f, r) \in p$, and so $p \in \overline{e(X(L_s f, r))}$. Therefore, $|(T_p f)(s)| \leq r$, as required. $\square$

Theorem 1. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $C = \overline{\varphi}$. The following statements are equivalent:

(i) C is a subsemigroup of δS.
(ii) If $f \in I$ and $p \in C$, then $T_p f \in I$.
(iii) If $A \in \varphi$ and $p \in C$, then $\Omega_p(A) \in \varphi$.

Proof. (i) $\Rightarrow$ (ii) Suppose that C is a subsemigroup of δS. Let $f \in I$ and let $p \in C$. By Lemma 3 and [2, Theorem 13], the function $\widehat{T_p f}$ vanishes on C, and so $T_p f \in I$ by [2, Theorem 13].

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Let $A \in \varphi$ and let $p \in C$. Pick $f \in I$ and $r > 0$ such that $X(f,r) \subseteq A$. Then $X(T_p f, r/2) \subseteq \Omega_p(A)$ by Lemma 4. Since $X(T_p f, r/2) \in \varphi$ by assumption, the statement follows.

(iii) $\Rightarrow$ (i) Suppose that (iii) holds. Let $p, q \in C$. If $A \in \varphi$, then $A^\circ \in \varphi$, so $\Omega_q(A^\circ) \in \varphi$ by assumption, and so $\Omega_q(A^\circ) \in p$. Therefore, $pq \in \mathrm{cl}_{\delta S}(\widehat{A^\circ})$ by Lemma 2 (iv), and so $pq \in \overline{e(A)}$ by [2, Lemma 4 (ii)], as required. $\square$

An application of the previous theorem to a single element p of δS implies the following corollary. The content of the following corollary is exactly the same as in the case that S is discrete and $\delta S = \beta S$ (see [5, p. 76]).

Corollary 2. If $p \in \delta S$, then the following statements are equivalent:

(i) p is an idempotent.
(ii) $\Omega_p(A) \in p$ for every $A \in p$.

Next, we proceed to characterize left, right, and two-sided ideals of δS.

Definition 2. An $\mathcal{F}$-filter φ on S is *left* [*right*] *thick* if and only if, for every $A \in \varphi$ and for every $s \in S$, there exists a set $B \in \varphi$ such that $sB \subseteq A$ [$Bs \subseteq A$].

Note that an $\mathcal{F}$-filter φ on S is left thick if and only if $\Omega_\varphi(A) = S$ for every $A \in \varphi$. If S is algebraically a group, then an $\mathcal{F}$-filter φ on S is left [right] thick if and only if $sA \in \varphi$ [$As \in \varphi$] for every $A \in \varphi$ and for every $s \in S$.

Theorem 2. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $L = \overline{\varphi}$. The following statements are equivalent:

(i) L is a left ideal of δS.
(ii) I is left translation invariant.
(iii) φ is left thick.

Proof. (i) $\Rightarrow$ (ii) As in the previous proof, the function $\widehat{L_s f}$ vanishes on L for every $f \in \mathcal{F}$ and for every $s \in S$.

(ii) $\Rightarrow$ (iii) This follows from the equality $X(L_s f, r) = \lambda_s^{-1}(X(f,r))$.

(iii) $\Rightarrow$ (i) Suppose that φ is left thick. It is enough to show that $e(s)q \in L$ for every $s \in S$ and for every $q \in L$. So, let $s \in S$, let $q \in L$, and let $A \in \varphi$. Pick a set $B \in \varphi$ such that $sB \subseteq A$. Since $q \in \overline{e(B)}$, we have $e(s)q \in \overline{e(A)}$, as required. $\square$

Similar arguments as given in the proof of Theorem 1 apply to prove the next theorem.

Theorem 3. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $R = \overline{\varphi}$. The following statements are equivalent:

(i) R is a right ideal of δS.
(ii) If $f \in I$ and $p \in \delta S$, then $T_p f \in I$.
(iii) If $A \in \varphi$ and $p \in \delta S$, then $\Omega_p(A) \in \varphi$.

In particular, statement (ii) of the previous theorem implies that $T_{e(s)} f \in I$ for every $f \in I$ and for every $s \in S$, and so I is right translation invariant. The equality $X(R_s f, r) = \rho_s^{-1}(X(f,r))$ implies that φ is right thick. However, this is not enough to ensure that R is a right ideal of δS. If S is commutative, then an $\mathcal{F}$-filter φ on S is left thick if and only if φ is right thick. But, in general, a closed left ideal of δS need not be a right ideal. For the following theorem, recall that δS is a semitopological semigroup if and only if $\mathcal{F} \subseteq \mathcal{WAP}(S)$ (see [4, pp. 138, 143]). We leave the proof to the reader.

Theorem 4. Suppose that $\mathcal{F} \subseteq \mathcal{WAP}(S)$. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $R = \overline{\varphi}$. The following statements are equivalent:

(i) R is a right ideal of δS.
(ii) I is right translation invariant.
(iii) φ is right thick.

Combining Theorem 2 and Theorem 3, we obtain the following corollary.

Corollary 3. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $J = \overline{\varphi}$. The following statements are equivalent:

(i) J is an ideal of δS.
(ii) I is left translation invariant and $T_p f \in I$ for every $f \in I$ and for every $p \in \delta S$.
(iii) φ is left thick and $\Omega_p(A) \in \varphi$ for every $A \in \varphi$ and for every $p \in \delta S$.

If S is commutative, then the description of ideals of δS is simpler.

Theorem 5. Suppose that S is commutative. Let φ be an $\mathcal{F}$-filter on S, let I be the ideal of $\mathcal{F}$ such that φ is generated by $\mathcal{B}(I)$, and let $J = \overline{\varphi}$. The following statements are equivalent:

(i) J is an ideal of δS.
(ii) If $f \in I$ and $p \in \delta S$, then $T_p f \in \mathcal{F}$.
(iii) If $A \in \varphi$ and $p \in \delta S$, then $\Omega_p(A) \in \varphi$.

Proof. We need only to show that statement (iii) implies that φ is left thick. But if (iii) holds, then J is a right ideal of δS by Theorem 3, so φ is right thick (see the paragraph after Theorem 3), and so φ is left thick. $\square$

5. Minimal Ideal of δS and its closure

In this section, we assume that S is embedded in δS. We shall consider S as a subspace of δS, and so we denote $e(s)$ simply by s for every $s \in S$. Furthermore, we assume the following:

(1) If $A \subseteq S$, $s \in S$, and $p \in \delta S$, then $A \in sp$ if and only if $s^{-1}A \in p$.

Recall that (1) is satisfied if S is algebraically a group. Also, (1) is satisfied for the Stone-Čech compactification of a discrete semigroup. Under these assumptions, we characterize those points of δS which are in the minimal ideal K of δS or in its closure. In what follows, we apply the fact that K is the union of all minimal left ideals of δS (see [5, p.34]).

Our assumptions and the continuity of ρ_p imply the following statement.

Lemma 5. If $A \subseteq S$ and $p \in \delta S$, then $\Omega_p(A)$ is an open subset of S.

Definition 3. A subset A of S is *syndetic* if and only if there exists a finite subset F of S such that $S = \bigcup_{s\in F} s^{-1}A$.

Theorem 6. If $p \in \delta S$, then the following statements are equivalent:

(i) $p \in K$.
(ii) If $A \in p$, then $\Omega_p(A)$ is syndetic.
(iii) If $q \in \delta S$, then $p \in (\delta S)qp$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $p \in K$. Let $A \in p$. Let L be the minimal left ideal of δS such that $p \in L$. If $q \in L$, then $L = (\delta S)q = \overline{Sq}$, and so $\widehat{A} \cap Sq \neq \emptyset$. Pick some $s \in S$ such that $sq \in \widehat{A}$. Then $s^{-1}A \in q$. So, for every $q \in L$, there exists some $s \in S$ such that $s^{-1}A \in q$. Therefore, $L \subseteq \bigcup_{s\in F} \widehat{s^{-1}A}$ for some finite subset F of S.

To see that $\Omega_p(A)$ is syndetic, let $t \in S$. Since $tp \in L$, there exists some $s \in F$ such that $s^{-1}A \in tp$. Then $t^{-1}s^{-1}A = (st)^{-1}A \in p$, so $st \in \Omega_p(A)$, and so $t \in s^{-1}\Omega_p(A)$ for some $s \in F$, as required.

(ii) $\Rightarrow$ (iii) Suppose that (ii) holds. Suppose also that there exists some $q \in \delta S$ such that $p \notin (\delta S)qp$. Since δS is a regular topological space, there exists a set $A \in p$ such that $\mathrm{cl}_{\delta S}(\widehat{A}) \cap (\delta S)pq = \emptyset$. By assumption, there exists a finite subset F of S such that $S = \bigcup_{t\in F} t^{-1}\Omega_p(A)$. Pick $t \in F$ such that $q \in \overline{t^{-1}\Omega_p(A)}$. By Lemma 5 and [2, Lemma 4 (ii)], we have $q \in \mathrm{cl}_{\delta S}(\widehat{t^{-1}\Omega_p(A)})$, and so $tq \in \mathrm{cl}_{\delta S}(\widehat{\Omega_p(A)})$ by Lemma 2 (ii). Therefore, $tqp \in \mathrm{cl}_{\delta S}(\widehat{A})$ by Lemma 2 (iv), a contradiction.

(iii) $\Rightarrow$ (i) Choose any $q \in K$. $\square$

Definition 4. A subset A of S is *piecewise syndetic* if and only if there exists a finite subset F of S such that the family $\{s^{-1}(\bigcup_{t\in F} t^{-1}A) : s \in S\}$ has the finite intersection property.

Theorem 7. If $A \subseteq S$, then the following statements hold:

(i) If $\widehat{A} \cap K \neq \emptyset$, then A is piecewise syndetic.
(ii) If A is open and piecewise syndetic, then $\mathrm{cl}_{\delta S}(\widehat{A}) \cap K \neq \emptyset$.

Proof. (i) Suppose that $p \in \widehat{A} \cap K$ for some $p \in \delta S$. Then $\Omega_p(A)$ is syndetic by Theorem 6. Let F be a finite subset of S such that $S = \bigcup_{t\in F} t^{-1}\Omega_p(A)$. Let $s \in S$ and pick $t \in F$ with $s \in t^{-1}\Omega_p(A)$. Then $s^{-1}(t^{-1}A) \in p$. Therefore, $s^{-1}(\bigcup_{t\in F} t^{-1}A) \in p$ for every $s \in S$, and so A is piecewise syndetic.

(ii) Suppose that A is open and piecewise syndetic. Pick a finite subset F of S such that the family $\{s^{-1}(\bigcup_{t\in F} t^{-1}A) : s \in S\}$ has the finite intersection property. Put $B = \bigcup_{t\in F} t^{-1}A$ and

$$\mathcal{A} = \{X(f,r) : f \in \mathcal{Z}(s^{-1}B),\, r > 0,\, s \in S\}.$$

Then B is an open subset of S. By [2, Lemma 1], the family $\mathcal{A}$ is an $\mathcal{F}$-family on S such that $\mathcal{A}$ has the finite intersection property, and so there exists some $p \in \delta S$ such that $\mathcal{A} \subseteq p$ by [2, Lemma 2].

For a while, fix $s \in S$. Since $X(f,r) \in p$ for every $f \in \mathcal{Z}(s^{-1}B)$ and for every $r > 0$, we have $p \in \overline{s^{-1}B}$ by [2, Lemma 3], and so $p \in \mathrm{cl}_{\delta S}(\widehat{s^{-1}B})$ by [2, Lemma 4 (ii)]. Therefore, $sp \in \mathrm{cl}_{\delta S}(\widehat{B})$ by Lemma 2 (ii). Since $s \in S$ was arbitrary, we have $(\delta S)p \subseteq \mathrm{cl}_{\delta S}(\widehat{B})$. Here, $(\delta S)p$ is a left ideal of δS, and so we may pick some element $q \in K \cap (\delta S)p$. Since $q \in \mathrm{cl}_{\delta S}(\widehat{B})$, we have $B \cap C \neq \emptyset$ for every $C \in q$. By the definition of the set B, we may assume that there exists an element $t \in F$ such that $t^{-1}A \cap C \neq \emptyset$ for every $C \in q$. Then $q \in \overline{t^{-1}A}$ by [2, Lemma 3], so $q \in \mathrm{cl}_{\delta S}(\widehat{t^{-1}A})$ by [2, Lemma 4 (ii)], and so $tq \in \mathrm{cl}_{\delta S}(\widehat{A})$ by Lemma 2 (ii). Since $q \in K$, we have $tq \in K$, and so $K \cap \mathrm{cl}_{\delta S}(\widehat{A}) \neq \emptyset$. □

Corollary 4. If $p \in \delta S$, then $p \in \overline{K}$ if and only if every $A \in p$ is piecewise syndetic.

Proof. Necessity follows from Theorem 7 (i). Sufficiency follows from Theorem 7 (ii), since δS is a regular topological space. □

Definition 5. A subset A of S is *central* if and only if there exists an idempotent $e \in K$ such that $A \in e$.

Theorem 8. Let $A \subseteq S$. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) hold for the following statements:

(i) $\widehat{A} \cap K \neq \emptyset$.
(ii) The set $\{s \in S : s^{-1}A \text{ is central}\}$ is syndetic.
(iii) There exists some $s \in S$ such that $s^{-1}A$ is central.
(iv) A is piecewise syndetic.

Proof. (i) ⇒ (ii) Suppose that $p \in \widehat{A} \cap K$ for some $p \in \delta S$. Let L be the minimal left ideal of δS such that $p \in L$. Pick an idempotent $e \in L$. Then $A \in p = pe$, so $\Omega_e(A) \in p$ by Lemma 2 (iii), and so there exists some $s \in S$ such that $s^{-1}A \in e$. Since $e \in K$, the set $B = \Omega_e(s^{-1}A)$ is syndetic by Theorem 6. Pick a finite subset F of S such that $S = \bigcup_{t\in F} t^{-1}B$. Put $C = \{s \in S : s^{-1}A \text{ is central}\}$. Now, it is enough to show that $S = \bigcup_{u\in sF} u^{-1}C$. But if $v \in S$, then there exists an element $t \in F$ such that $tv \in B$, so

$(tv)^{-1}s^{-1}A = (stv)^{-1}A \in e$, and so $stv \in C$. Therefore, $v \in (st)^{-1}C$, as required.

(ii) $\Rightarrow$ (iii) A syndetic subset of S is not empty.

(iii) $\Rightarrow$ (iv) Suppose that there exists some $s \in S$ such that $s^{-1}A$ is central. Pick an idempotent $e \in K$ such that $s^{-1}A \in e$. Then $A \in se$. Since $se \in K$, the set A is piecewise syndetic by Theorem 7 (i) $\square$

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